

On C^1 -class local diffeomorphisms whose periodic points are nonuniformly expanding[☆]

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Abstract

Using a sifting-shadowing combination, we prove in this paper that an arbitrary C^1 -class local diffeomorphism f of a closed manifold M^n is uniformly expanding on the closure $\text{Cl}_{M^n}(\text{Per}(f))$ of its periodic point set $\text{Per}(f)$, if it is nonuniformly expanding on $\text{Per}(f)$.

Keywords: Nonuniformly expanding maps, sifting-shadowing combination

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1. Introduction

We consider a discrete-time differentiable semi-dynamical system

$$f: M^n \rightarrow M^n$$

which is a C^1 -class local diffeomorphism of a closed manifold M^n , where $n \geq 1$; that is to say, f is surjective and for any $x \in M^n$, there is an open neighborhood U_x around x in M^n such that f is C^1 -class diffeomorphic restricted to U_x .

1.1. Motivation

A point $p \in M^n$ is said to be *periodic with period* $\tau \geq 1$, if $f^\tau(p) = p$. For a number of situations in smooth ergodic theory and differentiable dynamical systems, the “nonuniform hyperbolicity” of its periodic point set, written as $\text{Per}(f)$, is often proven or assumed; for example, see the classical works [14, 22, 23, 25, 3, 17, 16]. Then, extending the hyperbolicity from the periodic points to the whole manifold or the closure of $\text{Per}(f)$ is a deep and important problem.

In this paper, we are concerned with the study of conditions for a nonuniformly expanding endomorphism to be uniformly expanding. There have been a few results concerning this. One of these results is the remarkable Theorem A of Mañé [24] for $C^{1+\text{Hölder}}$ endomorphisms of the unit circle \mathbb{T}^1 ; some interesting other results for C^1 -class local diffeomorphisms of M^n where $n \geq 2$, have appeared in several recent papers [1, 6, 7, 8, 10].

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1.2. Basic concepts

Before we pursue a further discussion, let us first recall some basic concepts. As usual, by $\text{Diff}_{\text{loc}}^1(M^n)$ we denote the set of all C^1 -class local diffeomorphisms of the closed manifold M^n , equipped with the usual C^1 -topology.

By an abuse of notation, $\|\cdot\|_{\text{co}}$ means the co-norm (also called minimum norm) defined in the following way: for any $f \in \text{Diff}_{\text{loc}}^1(M^n)$,

$$\|D_x f^\ell\|_{\text{co}} = \min_{v \in T_x M^n, \|v\|=1} \|(D_x f^\ell)v\|$$

for the derivatives $D_x f^\ell: T_x M^n \rightarrow T_{f^\ell(x)} M^n$ for all $x \in M^n$ and $\ell \geq 1$. Since f is locally diffeomorphic, $\|D_x f^\ell\|_{\text{co}} = \|(D_x f^\ell)^{-1}\|^{-1} > 0$. On the other hand, we have

$$\|D_x f^{\ell+m}\|_{\text{co}} \geq \|(D_x f^\ell)\|_{\text{co}} \cdot \|(D_{f^\ell(x)} f^m)\|_{\text{co}}$$

for any $x \in M^n$ and any $\ell, m \geq 1$.

For any point $x \in M^n$, let

$$\lambda_{\min}(x, f) = \limsup_{\ell \rightarrow +\infty} \frac{1}{\ell} \log \|D_x f^\ell\|_{\text{co}}$$

be the *minimal Lyapunov exponent* of f at the base point x . Our question considered here is this: If $\lambda_{\min}(x, f) > 0$ for all $x \in \text{Per}(f)$, whether f is expanding on the closure of $\text{Per}(f)$. Let us first see an example. We denote by $\{0, 1\}^{\mathbb{N}}$ the compact topological space of all the one-sided infinite sequences $i: \mathbb{N} \rightarrow \{0, 1\}$ and let $S(\alpha, \gamma) = \{S_0, S_1\}$ where

$$S_0 = \alpha \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad S_1 = \gamma \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad (\alpha, \gamma > 1).$$

Based on [5], there exists a pair of α, γ such that for every periodic sequence $i \in \{0, 1\}^{\mathbb{N}}$

$$\lambda_{\min}(i, S(\alpha, \gamma)) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|S_{i_1} \cdots S_{i_n}\|_{\text{co}} > 0,$$

but the linear cocycle, associated to $S(\alpha, \gamma)$ and driven by the Markov shift $\theta: i \mapsto i_{+1}$, is not expanding on $\{0, 1\}^{\mathbb{N}}$, although all the periodic sequences i form a dense subset of $\{0, 1\}^{\mathbb{N}}$.

This example motivates us to have to strengthen condition for uniformly expanding. The basic condition that we study in this paper is described as follows.

Definition. We say that $f \in \text{Diff}_{\text{loc}}^1(M^n)$ is *nonuniformly expanding* on a subset $\Lambda \subseteq M^n$ not necessarily closed, if there can be found a number $\lambda > 0$ such that

$$\limsup_{k \rightarrow +\infty} \frac{1}{k} \sum_{i=0}^{k-1} \log \|D_{f^i(x)} f\|_{\text{co}} \geq \lambda$$

for all $x \in \Lambda$. Here the constant λ is called an *expansion indicator* of f at Λ .

This is similar to what Alves, Bonatti and Viana has defined in [2]. Recall that for an arbitrary ergodic measure μ of f , based on the Kingman subadditive ergodic theorem [19] one can introduce the *minimal Lyapunov exponent of f restricted to μ* by

$$\lambda_{\min}(\mu, f) = \lim_{\ell \rightarrow +\infty} \frac{1}{\ell} \log \|D_x f^\ell\|_{\text{co}} \quad \mu\text{-a.e. } x \in M^n.$$

It is worthwhile noting here that restricted to a subset Λ , the nonuniformly expanding property of f is more stronger than the condition that f has only Lyapunov exponents $\lambda_{\min}(\mu, f)$, which are positive and uniformly bounded away from zero, for all ergodic measures μ in Λ . So, if f is nonuniformly expanding on Λ , then every ergodic measure μ of f distributed in Λ has only positive Lyapunov exponent $\lambda_{\min}(\mu, f)$. But, because here Λ is not necessarily a closed subset of M^n such as $\Lambda = \text{Per}(f)$, it is possible that there exists some ergodic measure μ which is just supported on the boundary $\partial\Lambda$, not in Λ , and which cannot be a-priori approximated arbitrarily by periodic measures even in the case $\Lambda = \text{Per}(f)$. This prevents us from using the expanding criteria already developed, for example, in [1, 6] and [25, Lemma I-5], to prove that f is uniformly expanding on the closure $\text{Cl}_{M^n}(\Lambda)$ of Λ in M^n .

1.3. Main results and outlines

Our principal result obtained in this paper can be stated as follows, which will be proved in Section 5.1 based on a series of lemmas developed in Sections 2, 3 and 4.

Theorem 1. *Let $f: M^n \rightarrow M^n$ be a C^1 -class local diffeomorphism on the closed manifold M^n , which is nonuniformly expanding on its periodic point set $\text{Per}(f)$. Then, there hold the following statements.*

- (1) *f is uniformly expanding on the closure $\text{Cl}_{M^n}(\text{Per}(f))$, i.e., there can be found two numbers $C > 0$ and $\lambda > 0$ such that*

$$\|(D_x f^k)v\| \geq C\|v\| \exp(k\lambda)$$

for all $v \in T_x M^n$, $x \in \text{Cl}_{M^n}(\text{Per}(f))$ and $k \geq 1$.

- (2) *For an arbitrary ergodic measure μ of f , either $\lambda_{\min}(\mu, f) \leq 0$, or $\lambda_{\min}(\mu, f) \geq \lambda$ and $\text{supp}(\mu) \subseteq \text{Cl}_{M^n}(\text{Per}(f))$.*

- (3) *If additionally $\text{Per}(f)$ is dense in the nonwandering point set $\Omega(f)$ of f , then f is uniformly expanding on M^n .*

The statement (1) of Theorem 1 is closely related to an important theorem of Mañé [25, Theorem II-1], which essentially reads as follows: If $f \in \text{Diff}^1(M^n)$ preserves a homogeneous dominated splitting $T_\Delta M^n = E \oplus F$ where $\Delta = \text{Cl}_{M^n}(\text{Per}(f))$, such that the bundle E is contracted by Df and at every periodic point p ,

$$\limsup_{k \rightarrow +\infty} \frac{1}{k} \sum_{i=0}^{k-1} \log \|(D_{f^i(p)} f)|F\|_{\text{co}} \geq \lambda$$

for some uniform constant $\lambda > 0$, then f is (uniformly) expanding along F on Δ . However, Mañé's theorem does not apply directly to our situation studied here, since f is not a diffeomorphism. In addition, the statement (1) of Theorem 1 is proved by Castro, Oliveira and Pinheiro [8] in the special case where f possesses the closing by periodic orbits property, and by Sun and Tian [28] in the generic case.

For any $f \in \text{Diff}_{\text{loc}}^1(M^n)$, by definition, the nonuniformly expanding for f on $\text{Per}(f)$ is equivalent to the property that there exists a constant $\lambda > 0$ such that

$$\int_{M^n} \log \|D_x f\|_{\text{co}} \mu(dx) \geq \lambda$$

for all ergodic measures μ of f supported on periodic orbits. However, from R. Mañé [25, Lemma I-5] it follows that, f is (uniformly) expanding on $\text{Cl}_{M^n}(\text{Per}(f))$ if and only if there exists an integer $m \geq 1$ and a constant $\lambda' > 0$ such that

$$\int_{M^n} \log \|D_x f^m\|_{\text{co}} \mu(dx) \geq \lambda'$$

for all ergodic measures μ of f supported on $\text{Cl}_{M^n}(\text{Per}(f))$. Since $\text{Per}(f)$ does not need to be closed in M^n and there is no a-priori generic condition, like closing by periodic orbits property, for the restriction of f to $\text{Cl}_{M^n}(\text{Per}(f))$ to ensure that each ergodic measure of f distributed on $\text{Cl}_{M^n}(\text{Per}(f))$ can be arbitrarily approximated by periodic ones, Mañé's criterion above does not work here. We will prove the uniformly expanding property by employing a Liaowise “sifting-shadowing combination” motivated by S.-T. Liao [22] and R. Mañé [25].

To overcome the non-invertibility of f , we will introduce the natural extension of f in Section 3. The idea of the proof of Theorem 1 is that if f had not been uniformly expanding on $\text{Cl}_{M^n}(\text{Per}(f))$ then, using the natural extension of f and a sifting lemma (Pliss lemma), we would construct an “abnormal” quasi-expanding pseudo-orbit string of f in $\text{Cl}_{M^n}(\text{Per}(f))$. Further, a shadowing lemma (Theorem 2.1) enables us to find an “abnormal” periodic orbit P whose minimal Lyapunov exponent $\lambda_{\min}(P, f)$ can approach arbitrarily to zero (Theorem 4.1), which contradicts the nonuniformly expanding property of f on $\text{Per}(f)$. As our sifting-shadowing combination, here we use the Pliss lemma (Lemma 2.2) and the shadowing lemma (Theorem 2.1) proved in Appendix in Section 6.

In the context of the stability conjecture of Palis and Smale, Pliss [27], Liao [22] and Mañé [23] were independently led to the notion of dominated splitting of the tangent bundle into two subbundles: one of them is definitely more contracted (or less expanded) than the other, after a uniform number of iterates. Recall from [22, 4] that for any $f \in \text{Diff}_{\text{loc}}^1(M^n)$ and $\eta > 0$, we say f has an “ $(\eta, 1)$ -dominated splitting” over $\text{Cl}_{M^n}(\text{Per}(f))$, provided that there exists a constant $C > 0$ and Df -invariant decomposition of $T_{\text{Cl}_{M^n}(\text{Per}(f))}M^n$ into two subbundles

$$T_x M^n = E(x) \oplus F(x) \quad \text{with} \quad \dim E(x) = 1 \quad \forall x \in \text{Cl}_{M^n}(\text{Per}(f))$$

such that

$$\frac{\|D_x f^k|E(x)\|}{\|D_x f^k|F(x)\|_{\text{co}}} \leq C \exp(-2\eta k) \quad \forall k \geq 1.$$

By choosing an adapted norm, there is no loss of generality in assuming $C = 1$ for simplicity.

As a result of Theorem 1, we will obtain the following statement in Section 5.2.

Theorem 2. *Let $f: M^n \rightarrow M^n$ be a C^1 -class local diffeomorphism where $n \geq 2$, and assume f possesses an $(\eta, 1)$ -dominated splitting over $\text{Cl}_{M^n}(\text{Per}(f))$. If every $p \in \text{Per}(f)$ have only positive Lyapunov exponents and such exponents are uniformly bounded away from 0, then f is uniformly expanding on $\text{Cl}_{M^n}(\text{Per}(f))$.*

Finally, in Section 5.3, we will apply Theorem 1 stated above to a C^1 -class local diffeomorphism of the circle \mathbb{T}^1 ; see Theorem 3 below.

2. Closing property of recurrent quasi-expanding orbit strings

To apply a sifting-shadowing combination, we need first to introduce a suitable shadowing lemma and a sifting lemma for local diffeomorphisms of the closed manifold M^n . For that, we have to introduce two notions: “ λ -quasi-expanding orbit-string” and “shadowing property” of quasi-expanding pseudo-orbit.

2.1. Closing up quasi-expanding strings

Consider an arbitrary $f \in \text{Diff}_{\text{loc}}^1(M^n)$. Recall that for any $\lambda > 0$, an ordered segment of orbit of f of length k

$$(x, f^k(x)) := (x, f(x), \dots, f^k(x)) \quad (k \geq 1)$$

is called a λ -quasi-expanding orbit-string of f if

$$\frac{1}{\ell} \sum_{j=1}^{\ell} \log \|D_{f^{k-j}(x)} f\|_{\text{co}} \geq \lambda \quad \forall \ell = 1, \dots, k.$$

As a complement to Liao's shadowing lemma [21], we could obtain the following shadowing lemma.

Theorem 2.1. *Given any $f \in \text{Diff}_{\text{loc}}^1(M^n)$ and any two numbers $\varepsilon > 0$ and $\lambda > 0$, there exists a number $\delta = \delta(\varepsilon, \lambda) > 0$ such that, if all $(x_i, f^{n_i}(x_i))$, $i = 0, \dots, k$, are λ -quasi-expanding orbit-strings satisfying $d(f^{n_i}(x_i), x_{i+1}) < \delta$ for all $0 \leq i \leq k$ where $x_{k+1} = x_0$, then there can be found a periodic point \mathbf{x} of f with period $\tau_{\mathbf{x}} = n_0 + \dots + n_k$ verifying*

$$d(f^{n_{-1} + \dots + n_{i-1} + j}(\mathbf{x}), f^j(x_i)) < \varepsilon \quad 0 \leq j \leq n_i, \quad 0 \leq i < k$$

and

$$\frac{1}{\ell} \sum_{j=1}^{\ell} \log \|D_{f^{\tau_{\mathbf{x}}-j}(\mathbf{x})} f\|_{\text{co}} \geq \lambda - \varepsilon \quad \forall \ell = 1, \dots, \tau_{\mathbf{x}}.$$

Here $n_{-1} = 0$ and $d(\cdot, \cdot)$ is an arbitrarily preassigned natural metric on M^n .

In fact, following the ideas of [18, 11], one can further obtain an (ε, ρ) -exponential closing property under this situation. Here the proof of this theorem is standard following [15]; see Appendix below for the details.

2.2. The Pliss lemma

For our sifting lemma, we shall apply the following reformulation of a result due to V. Pliss.

Lemma 2.2 ([27]). *Let $H > 0$ be arbitrarily given. For any $\gamma > \gamma' > 0$, there exists an integer $N_{\gamma, \gamma'} \geq 1$ and a real number $c_{\gamma, \gamma'} \in (0, 1)$ such that, if (a_0, \dots, a_{m-1}) with $m \geq N_{\gamma, \gamma'}$ and $|a_i| \leq H$ for all $0 \leq i < m$, is a " γ -string" in the sense that*

$$\frac{1}{m} \sum_{i=0}^{m-1} a_i \geq \gamma,$$

then there can be found integers $0 < n_1 < \dots < n_k \leq m$ with $k \geq \max\{1, mc_{\gamma, \gamma'}\}$ such that (a_0, \dots, a_{n_i-1}) is a " γ' -quasi-expanding string" for all $1 \leq i \leq k$, i.e.,

$$\frac{1}{J} \sum_{j=1}^J a_{n_i-j} \geq \gamma' \quad \forall J = 1, \dots, n_i.$$

We note here that in the above Pliss lemma, the numbers $N_{\gamma, \gamma'}$ and $c_{\gamma, \gamma'}$ both depend on the preassigned constant H . For our applications later, we will consider the special case where

$$H = \max \{ |\log \|D_x f\|_{\text{co}}|; x \in M^n \} \quad \text{and} \quad a_i = \log \|D_{f^i(x)} f\|_{\text{co}}$$

for a local diffeomorphism f and $x \in M^n$.

By a so-called *sifting-shadowing combination*, we mean a combinatorial application of a sifting lemma like Lemma 2.2 and a shadowing lemma like Theorem 2.1. It is an effective strategy to prove hyperbolicity in differentiable dynamical systems, see [22, 16, 12] for example.

2.3. Existence of periodic repellers

Using Theorem 2.1 and Lemma 2.2 for a C^1 -class local diffeomorphism f , we can obtain the following theorem on existence of periodic repellers under the assumption that f preserves an expanding ergodic measure. This theorem will be needed in the proof of the statement (2) of Theorem 1.

Theorem 2.3. *Let $f \in \text{Diff}_{\text{loc}}^1(M^n)$ preserve an ergodic probability measure μ . If the minimal Lyapunov exponent of f restricted to μ*

$$\lambda_{\min}(\mu, f) = \lim_{\ell \rightarrow +\infty} \frac{1}{\ell} \log \|D_x f^\ell\|_{\text{co}} > 0 \quad \mu\text{-a.e. } x \in M^n,$$

then for any $0 < \gamma'' < \lambda_{\min}(\mu, f)$, there exists a sequence of periodic repellers $\{P_\ell\}_1^\infty$ of f with

$$\lim_{\ell \rightarrow +\infty} P_\ell = \text{supp}(\mu)$$

in the sense of the Hausdorff topology, such that $\bigcup_\ell P_\ell$ is nonuniformly expanding for f^κ with an expansion indicator $\gamma''\kappa$, for some $\kappa \geq 1$.

We notice here that, if $f \in \text{Diff}^1(M^n)$ preserves an ergodic probability measure μ satisfying

$$\lambda_{\max}(\mu, f) = \lim_{\ell \rightarrow +\infty} \frac{1}{\ell} \log \|D_x f^\ell\| < 0 \quad \mu\text{-a.e. } x \in M^n,$$

then Liao proved, using his theory of standard systems of equations in [20], that μ is supported on a periodic attractor of f . Our Theorem 2.3 is thus an extension of Liao's result.

To prove this theorem, we need the following subadditive version of [13, Theorem 2], which guarantees the existence of a long γ -string. This long γ -string enables us to use the Pliss lemma (Lemma 2.2) and then the shadowing lemma (Theorem 2.1).

Lemma 2.4. *Let $\theta: X \rightarrow X$ be a discrete-time semidynamical system of a compact metrizable space X , which preserves a Borel probability measure μ , and $\{t_\ell\}_{\ell=1}^{+\infty}$ an integer sequence with*

$$t_1 \geq 1, \quad t_{\ell+1} = 2t_\ell \quad (\ell = 1, 2, \dots).$$

Let $\varphi: \mathbb{N} \times X \rightarrow \mathbb{R} \cup \{-\infty\}$ be a measurable function with the subadditivity property

$$\varphi(k_1 + k_2, x) \leq \varphi(k_1, x) + \varphi(k_2, \theta^{k_1}(x)) \quad \mu\text{-a.e. } x \in X$$

for any $k_1, k_2 \geq 1$, such that

(a) $\varphi(t, \cdot) \in \mathcal{L}^1(X, \mu)$ for all $t \geq 1$, and

(b) $\{t^{-1}\varphi(t, \cdot)\}_{t=1}^{+\infty}$ is bounded below by an μ -integrable function.

Then, there exists a Borel subset of μ -measure 1, written as $\widehat{\Gamma}$, such that for all $x \in \widehat{\Gamma}$,

$$\bar{\varphi}(x) = \lim_{t \rightarrow +\infty} \frac{1}{t} \varphi(t, x) \quad \text{with } \bar{\varphi}(\theta^t(x)) = \bar{\varphi}(x) \quad \forall t > 0$$

and

$$\lim_{\ell \rightarrow +\infty} \left\{ \lim_{k \rightarrow +\infty} \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{t_\ell} \varphi(t_\ell, \theta^{j t_\ell}(x)) \right\} = \bar{\varphi}(x).$$

Note. Here $\bar{\varphi}(x)$ is defined by the Kingman subadditive ergodic theorem [19] such that

$$\int_X \bar{\varphi}(x) \mu(dx) = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_X \varphi(t, x) \mu(dx) = \inf_{t \geq 1} \frac{1}{t} \int_X \varphi(t, x) \mu(dx).$$

Since

$$|\bar{\varphi}(x)| = \lim_{t \rightarrow +\infty} \frac{1}{t} |\varphi(t, x)| \leq \lim_{t \rightarrow +\infty} \frac{1}{t} \sum_{i=0}^{t-1} |\varphi(1, \theta^i(x))| = \psi(x) \quad \mu\text{-a.e. } x \in X,$$

where $\psi \in \mathcal{L}^1(X, \mu)$ is defined by the Birkhoff ergodic theorem for $\theta: X \rightarrow X$ and $|\varphi(1, \cdot)|$, hence under our hypothesis we have $\bar{\varphi} \in \mathcal{L}^1(X, \mu)$.

Proof. The following argument is parallel to that of [13, Theorem 2]. According to the Kingman subadditive ergodic theorem, there is a Borel set $\Gamma' \subset X$ of μ -measure 1 and a measurable function $\bar{\varphi} \in \mathcal{L}^1(X, \mu)$ such that

$$\bar{\varphi}(x) = \lim_{t \rightarrow +\infty} \frac{1}{t} \varphi(t, x) \quad \text{with } \bar{\varphi}(\theta^t(x)) = \bar{\varphi}(x) \quad \forall x \in \Gamma'$$

and

$$\int_X \bar{\varphi}(x) \mu(dx) = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_X \varphi(t, x) \mu(dx).$$

So, for the given sequence $\{t_\ell\}_1^\infty$ we have

$$\lim_{\ell \rightarrow +\infty} \int_X \frac{1}{t_\ell} \varphi(t_\ell, \theta^\alpha(x)) \mu(dx) = \int_X \bar{\varphi}(x) \mu(dx)$$

uniformly for $\alpha \in \mathbb{Z}_+$, since μ is θ^α -invariant. For any $\varepsilon > 0$, there thus exists an $\ell(\varepsilon) > 0$ such that, if $\ell \geq \ell(\varepsilon)$ then

$$\frac{1}{k} \sum_{j=0}^{k-1} \int_X \left\{ \frac{1}{t_\ell} \varphi(t_\ell, \theta^{j t_\ell}(x)) - \bar{\varphi}(x) \right\} \mu(dx) \leq \frac{\varepsilon}{2} \quad \forall k \in \mathbb{N}.$$

This means that for all $\ell \geq \ell(\varepsilon)$ there holds the inequality

$$\int_X \frac{1}{k} \sum_{j=0}^{k-1} \left\{ \frac{1}{t_\ell} \varphi(t_\ell, \theta^{j t_\ell}(x)) - \bar{\varphi}(x) \right\} \mu(dx) \leq \frac{\varepsilon}{2} \quad \forall k \in \mathbb{N}.$$

For any $\ell \geq 1$, we now consider the sample $\theta^{\ell} : X \rightarrow X$ which also preserves μ , but not necessarily ergodic even if μ is ergodic under $\theta : X \rightarrow X$. From the Birkhoff ergodic theorem and the subadditivity of $\varphi(t, x)$, it follows that there is a θ^{ℓ} -invariant Borel subset $W_{\ell}^* \subset \Gamma'$ of μ -measure 1 such that

$$\lim_{k \rightarrow +\infty} \frac{1}{k} \sum_{j=0}^{k-1} \left\{ \frac{1}{t_{\ell}} \varphi(t_{\ell}, \theta^{j t_{\ell}}(x)) - \bar{\varphi}(x) \right\} = h_{\ell}^*(x) \geq 0 \quad \forall x \in W_{\ell}^*$$

for some $h_{\ell}^*(\cdot) \in \mathcal{L}^1(X, \mu)$, for all $\ell \geq 1$. Set

$$\widehat{\Gamma} = \bigcap_{\ell=1}^{+\infty} W_{\ell}^*.$$

Clearly, $\widehat{\Gamma} \subset \Gamma'$ and $\mu(\widehat{\Gamma}) = 1$. By **(b)**, the sequence of functions $\left\{ \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{t_{\ell}} \varphi(t_{\ell}, \theta^{j t_{\ell}}(\cdot)) \right\}_{k=1}^{+\infty}$ is bounded below by an μ -integrable function. Thus from Fatou's lemma, there follows that

$$\begin{aligned} \int_{\widehat{\Gamma}} h_{\ell}^*(x) \mu(dx) &\leq \liminf_{k \rightarrow +\infty} \int_{\widehat{\Gamma}} \frac{1}{k} \sum_{j=0}^{k-1} \left\{ \frac{1}{t_{\ell}} \varphi(t_{\ell}, \theta^{j t_{\ell}}(x)) - \bar{\varphi}(x) \right\} \mu(dx) \\ &\leq \frac{\varepsilon}{2} \end{aligned}$$

for all $\ell \geq \ell(\varepsilon)$, and hence

$$\lim_{\ell \rightarrow +\infty} \int_{\widehat{\Gamma}} h_{\ell}^*(x) \mu(dx) = 0.$$

So, one can find a subsequence $\{h_{\ell_k}^*(\cdot)\}_{k=1}^{+\infty}$ such that

$$h_{\ell_k}^*(x) \rightarrow 0 \quad \text{as } k \rightarrow +\infty \quad \text{for } \mu\text{-a.e. } x \in \widehat{\Gamma}.$$

In addition, noting $t_{\ell+1} = 2t_{\ell}$ for any $\ell \geq 1$, for all $x \in \widehat{\Gamma}$

$$\begin{aligned} h_{\ell}^*(x) &= \lim_{k \rightarrow +\infty} \frac{1}{2k} \sum_{j=0}^{2k-1} \left\{ \frac{1}{t_{\ell}} \varphi(t_{\ell}, \theta^{j t_{\ell}}(x)) - \bar{\varphi}(x) \right\} \\ &\geq \lim_{k \rightarrow +\infty} \frac{1}{k} \sum_{j=0}^{k-1} \left\{ \frac{1}{t_{\ell+1}} \varphi(t_{\ell+1}, \theta^{j t_{\ell+1}}(x)) - \bar{\varphi}(x) \right\} \\ &= h_{\ell+1}^*(x). \end{aligned}$$

This implies that

$$h_{\ell}^*(x) \rightarrow 0 \quad \text{as } \ell \rightarrow +\infty \quad \text{for } \mu\text{-a.e. } x \in \widehat{\Gamma},$$

which proves the lemma. \square

Now, we can readily prove Theorem 2.3 stated before based on Lemmas 2.4 and 2.2 and Theorem 2.1.

Proof of Theorem 2.3. Assume that μ is not supported on a periodic orbit of f ; otherwise the statement holds trivially. Let γ, γ' and γ'' be three constants such that

$$\lambda_{\min}(\mu, f) > \gamma > \gamma' > \gamma'' > 0,$$

and define the subadditive functions

$$\varphi(t, x) = -\log \|D_x f^t\|_{\text{co}} \quad \forall (t, x) \in \mathbb{N} \times M^n.$$

From Lemma 2.4 with $X = M^n$ and $\theta = f$, it follows that there is an $\ell > 0$ and a non-periodic point $y \in \text{supp}(\mu) \cap \widehat{\Gamma}$, where $\widehat{\Gamma}$ is defined by Lemma 2.4, such that

$$\lim_{k \rightarrow +\infty} \frac{1}{k} \sum_{j=0}^{k-1} \log \|D_{f^{j\ell}(y)} f^{t_\ell}\|_{\text{co}} > \gamma t_\ell \quad \text{and} \quad \mu = \lim_{J \rightarrow +\infty} \frac{1}{J} \sum_{j=0}^{J-1} \delta_{f^j(y)},$$

where δ_y denotes the Dirac measure at y and the integer t_ℓ is given as in Lemma 2.4.

From Lemma 2.2 with $a_j = \log \|D_{f^{j\ell}(y)} f^{t_\ell}\|_{\text{co}}$ for all $j \geq 0$, it follows that one can find a positive integer sequence $\{n_k\}_{k=1}^{+\infty}$ with $n_k \rightarrow +\infty$ such that (a_0, \dots, a_{n_k-1}) is a “ $\gamma' t_\ell$ -quasi-expanding string” for all $k \geq 1$, i.e.,

$$\frac{1}{m} \sum_{j=1}^m a_{n_k-j} \geq \gamma' t_\ell \quad \forall m = 1, \dots, n_k.$$

For the simplicity of notation, we assume $t_\ell = 1$; otherwise we consider f^{t_ℓ} instead of f when applying Theorem 2.1.

Write $x_j = f^j(y)$ for all $j \geq 0$. By the compactness of M^n , there can be found two subsequences $\{n'_k\}_{k=1}^{+\infty}$ and $\{n''_k\}_{k=1}^{+\infty}$ of $\{n_k\}_{k=1}^{+\infty}$ such that

$$\tau_k := n''_k - n'_k \rightarrow +\infty, \quad d(x_{n'_k}, x_{n''_k}) \rightarrow 0 \quad \text{as } k \rightarrow +\infty$$

and

$$\frac{n'_k}{\tau_k} \leq \frac{1}{2} \quad \forall k \geq 1.$$

Since

$$\mu_k := \frac{1}{\tau_k} \sum_{j=0}^{\tau_k-1} \delta_{x_{n'_k+j}} = \left(1 + \frac{n'_k}{\tau_k}\right) \frac{1}{n'_k} \sum_{j=0}^{n'_k-1} \delta_{x_j} - \frac{n'_k}{\tau_k} \frac{1}{n'_k} \sum_{j=0}^{n'_k-1} \delta_{x_j},$$

there is no loss of generality in assuming that μ_k converges weakly-* to μ as $k \rightarrow +\infty$. In fact, from

$$\frac{1}{n''_k} \sum_{j=0}^{n''_k-1} \delta_{x_j} \xrightarrow{\text{weakly-*}} \mu \quad \text{and} \quad \frac{1}{n'_k} \sum_{j=0}^{n'_k-1} \delta_{x_j} \xrightarrow{\text{weakly-*}} \mu \quad \text{as } k \rightarrow +\infty,$$

it follows that

$$\lim_{k \rightarrow +\infty} \int_{M^n} h(x) \mu_k(dx) = \lim_{k \rightarrow +\infty} \left(1 + \frac{n'_k}{\tau_k} - \frac{n'_k}{\tau_k}\right) \int_{M^n} h(x) \mu(dx) = \int_{M^n} h(x) \mu(dx)$$

for any $h \in C^0(M^n)$. Noting that $(x_{n'_k}, f^{\tau_k}(x_{n'_k}))$ is a γ' -quasi-expanding orbit-string of f with $d(x_{n'_k}, f^{\tau_k}(x_{n'_k}))$ converging to 0, and $x_{n''_k} = f^{\tau_k}(x_{n'_k})$.

Let $d_H(\cdot, \cdot)$ denote the Hausdorff metric for nonempty compact sets of M^n . Then, it follows, from Theorem 2.1 with $\lambda = \gamma'$ and $0 < \varepsilon < \gamma' - \gamma''$, that there can be found a sequence of periodic repellers $\{P_{k_j}\}$ of f with

$$\lim_{j \rightarrow +\infty} P_{k_j} = \Delta \subseteq \text{supp}(\mu) \quad \text{and} \quad \lim_{j \rightarrow +\infty} d_H(P_{k_j}, (x_{n'_{k_j}}, f^{\tau_{k_j}}(x_{n'_{k_j}}))) = 0,$$

and f is nonuniformly expanding on $\bigcup_{k_j} P_{k_j}$ with an expansion indicator $\lambda \geq \gamma''$. So, we can obtain that $\lim_{j \rightarrow +\infty} \text{supp}(\mu_{k_j}) = \Delta$.

It is clear that $\Delta = \text{supp}(\mu)$. In fact, if this fails, there is some $\hat{x} \in \text{supp}(\mu) \setminus \Delta$ and further let $d(\hat{x}, \Delta) = r > 0$. Then there is a continuous function $\xi: M^n \rightarrow [0, 1]$ satisfying

$$\xi(\hat{x}) = 1 \quad \text{and} \quad \xi(x) = 0 \quad \forall x \in M^n \text{ with } d(x, \Delta) \leq \frac{r}{2};$$

hence

$$0 < \int_{M^n} \xi(x) \mu(dx) = \lim_{j \rightarrow +\infty} \int_{M^n} \xi(x) \mu_{k_j}(dx) = \lim_{j \rightarrow +\infty} \int_{\text{supp}(\mu_{k_j})} \xi(x) \mu_{k_j}(dx) = 0,$$

a contradiction.

This ends the proof of Theorem 2.3. □

3. Natural extension of local diffeomorphisms

For a diffeomorphism $f: M^n \rightarrow M^n$, Mañé's arguments in [25] relies on the concept “ (t, γ) -set”. Let K be an f -invariant compact subset of M^n and $t \geq 1, \gamma > 0$. We say K to be a (t, γ) -set of f if for every $z \in K$ there exists an integer $m_z \in [0, t)$ such that

$$\frac{1}{\ell} \sum_{i=1}^{\ell} \log \|D_{f^{-i}(f^{m_z}(z))} f\|_{\text{co}} \geq \gamma \quad \forall \ell \geq 1.$$

Now, since f is not invertible, f^{-i} makes no sense here. To control the non-invertibility of f in the context of Theorem 1, we need to introduce the natural extension of f .

3.1. The extension of a cocycle

Let $f: X \rightarrow X$ be an arbitrarily given continuous endomorphism of a compact metric space (X, d) . Let

$$\Sigma_f = \{ \bar{x} = (\dots, x_i, \dots, x_{-1}, x_0) \in X^{\mathbb{Z}_-} \mid f(x_{i-1}) = x_i \quad \forall i \in \mathbb{Z}_- \},$$

where $\mathbb{Z}_- = \{0, -1, -2, \dots\}$. Then,

$$d_f(\bar{x}, \bar{y}) = \sum_{i=0}^{-\infty} 2^i \min\{1, d(x_i, y_i)\} \quad \forall \bar{x}, \bar{y} \in \Sigma_f$$

is a metric on Σ_f under which the shift mapping

$$\sigma_f: \Sigma_f \rightarrow \Sigma_f; \quad (\dots, x_i, \dots, x_{-1}, x_0) \mapsto (\dots, x_i, \dots, x_{-1}, x_0, f(x_0))$$

is a topological dynamical system (homeomorphism) on the compact metric space (Σ_f, d_f) . Let

$$\pi: \Sigma_f \rightarrow X; \quad (\dots, x_i, \dots, x_{-1}, x_0) \mapsto x_0$$

be the natural projection.

If f is a C^1 -class local diffeomorphism of the closed manifold $X = M^n$, we further set

$$T_{\bar{x}}\Sigma_f = T_{\pi(\bar{x})}M^n \quad \text{and} \quad F_{\bar{x}}: T_{\bar{x}}\Sigma_f \rightarrow T_{\sigma_f(\bar{x})}\Sigma_f; \quad v \mapsto (D_{\pi(\bar{x})}f)v \quad \forall \bar{x} \in \Sigma_f.$$

Then, we obtain the natural linear skew-product dynamical system

$$\begin{array}{ccc} T\Sigma_f & \xrightarrow{F} & T\Sigma_f \\ \text{Pr} \downarrow & & \downarrow \text{Pr} \\ \Sigma_f & \xrightarrow{\sigma_f} & \Sigma_f \end{array} \quad \text{where } F(\bar{x}, v) = (\sigma_f(\bar{x}), F_{\bar{x}}(v)) \text{ and } \text{Pr}: (\bar{x}, v) \mapsto \bar{x}.$$

Note here that the inverse of σ_f is defined as

$$\sigma_f^{-1}: \Sigma_f \rightarrow \Sigma_f; \quad (\dots, x_i, \dots, x_{-1}, x_0) \mapsto (\dots, x_i, \dots, x_{-1}).$$

For any forward invariant set Λ of f , let $\Lambda_f = \pi^{-1}(\Lambda)$, which is called the “extension” of Λ under f . It is easy to see that Λ_f is also a forward σ_f -invariant set, i.e., $\sigma_f(\Lambda_f) \subseteq \Lambda_f$.

The closure of a subset Y in a topological space Z is denoted by $\text{Cl}_Z(Y)$. By the continuity of π and the definition of $d_f(\cdot, \cdot)$, we can obtain that

$$\pi(\text{Cl}_{\Sigma_f}(\Lambda_f)) \subseteq \text{Cl}_X(\Lambda) \quad \text{and} \quad \pi(\Lambda_f) = \Lambda$$

for any subset $\Lambda \subseteq X$.

We will need the following lemma.

Lemma 3.1. *Let $f: X \rightarrow X$ be a continuous endomorphism of a compact metric space X . Then, for any forward f -invariant set $\Lambda \subseteq X$, there follows*

$$\pi(\text{Cl}_{\Sigma_f}(\Lambda_f)) = \text{Cl}_X(\Lambda),$$

where $\pi: \Sigma_f \rightarrow X$ is the natural projection.

Proof. Let Λ be a forward f -invariant subset of X . The statement trivially holds when $\Lambda = \emptyset$ or X . So, we now assume $\Lambda \neq \emptyset$ and $\neq X$. Let $x \in \text{Cl}_X(\Lambda) \setminus \Lambda$ be arbitrarily given. Then, there is a sequence of points p_j in Λ such that

$$p_j \rightarrow x \quad \text{as } j \rightarrow +\infty.$$

For all $j = 1, 2, \dots$, we arbitrarily pick

$$\bar{p}_j = (\dots, p_{j,i}, \dots, p_{j,-1}, p_{j,0}) \in \pi^{-1}(p_j) \subset \Lambda_f.$$

We now will define an $\bar{x} = (\dots, x_{-1}, x_0) = (x_i)_{i=0}^{-\infty} \in \text{Cl}_{\Sigma_f}(\Lambda_f)$ with $\pi(\bar{x}) = x$ by induction on i as follows:

First, let us choose $x_0 = x$. Obviously, $p_{j,0} \rightarrow x_0$ as $j \rightarrow +\infty$.

Secondly, since X is compact, we can pick a convergent subsequence $\left\{p_{j_k^{(1)}, -1}\right\}_{k=1}^{+\infty}$ from $\left\{p_{j, -1}\right\}_{j=1}^{+\infty}$. Let $x_{-1} = \lim_{k \rightarrow +\infty} p_{j_k^{(1)}, -1}$. It is easy to see that $f(x_{-1}) = x_0$ by the continuity of f . Assume $x_i, i \leq -2$, $\left\{j_k^{(-i)}\right\}_{k=1}^{\infty}$ have been defined such that

$$f(x_i) = x_{i+1} \quad \text{and} \quad p_{j_k^{(-i)}, i} \rightarrow x_i \quad \text{as } k \rightarrow +\infty.$$

By the compactness of the space X once again, we can pick a convergent subsequence, say $\left\{p_{j_k^{(-i+1)}, i-1}\right\}_{k=1}^{+\infty}$, from $\left\{p_{j, i-1}\right\}_{j=1}^{+\infty}$ such that $\left\{j_k^{(-i+1)}\right\}_{k=1}^{\infty} \subset \left\{j_k^{(-i)}\right\}_{k=1}^{\infty}$. We let $x_{i-1} = \lim_{k \rightarrow \infty} p_{j_k^{(-i+1)}, i-1}$. Then from the construction, we easily get $f(x_{i-1}) = x_i$. This completes the induction step.

Thus, we have chosen a point $\bar{x} = (\dots, x_i, \dots, x_1, x_0) \in \Sigma_f$ such that $\pi(\bar{x}) = x$ and $\bar{x} \in \text{Cl}_{\Sigma_f}(\Lambda_f)$. This implies that $\pi(\text{Cl}_{\Sigma_f}(\Lambda_f)) \supseteq \text{Cl}_X(\Lambda)$.

This shows Lemma 3.1. \square

We note that generally $\text{Cl}_X(\Lambda)_f$ is bigger than $\text{Cl}_{\Sigma_f}(\Lambda_f)$ when f is not injective, for an arbitrary f -invariant set $\Lambda \subset X$.

3.2. Obstruction and (t, γ) -set

Hereafter, let $f: M^n \rightarrow M^n$ be an arbitrarily given C^1 -class local diffeomorphism of the closed manifold M^n . Let $\sigma_f: \Sigma_f \rightarrow \Sigma_f$ and $F: T\Sigma_f \rightarrow T\Sigma_f$ be the natural extensions of f defined as in Section 3.1.

Definition 3.2. For $\bar{x} \in \Sigma_f$ and $m \geq 1$, $(\bar{x}, \sigma_f^m(\bar{x}))$ is called a ρ -string of F if

$$\frac{1}{m} \sum_{i=0}^{m-1} \log \|F_{\sigma_f^i(\bar{x})}\|_{\text{co}} \geq \rho.$$

Given any $n \geq 1$ and $\varrho > 0$, we say $(\bar{x}, \sigma_f^m(\bar{x}))$ is an (n, ϱ) -obstruction of F if $m \geq n$ and $(\bar{x}, \sigma_f^\ell(\bar{x}))$ is not a ϱ -string of F for all $n \leq \ell \leq m$, i.e.,

$$\frac{1}{\ell} \sum_{i=0}^{\ell-1} \log \|F_{\sigma_f^i(\bar{x})}\|_{\text{co}} < \varrho \quad \forall \ell \in [n, m].$$

Note here that $F_{\sigma_f^i(\bar{x})} = D_{f^i(\pi(\bar{x}))}f$ for any $i \geq 0$ from the definition of F in Section 3.1.

It is easily seen that if F is not expanding on a forward σ_f -invariant closed set $\Theta \subset \Sigma_f$, then to any $n \geq 1$ and $\varrho > 0$, from the compactness of Σ_f there can be found at least one $\bar{x} \in \Theta$ such that $(\bar{x}, \sigma_f^m(\bar{x}))$ is an (n, ϱ) -obstruction of F for all $m \geq n$.

Lemma 3.3 ([25, Lemma II-4]). Let $\bar{\gamma}_2 > \gamma_3 > 0$ and $(\bar{x}, \sigma_f^m(\bar{x}))$ be a $\bar{\gamma}_2$ -string of F , i.e.,

$$\frac{1}{m} \sum_{i=0}^{m-1} \log \|F_{\sigma_f^i(\bar{x})}\|_{\text{co}} \geq \bar{\gamma}_2.$$

Let $0 < n_1 < \dots < n_k \leq m$ be the set of integers such that $(\bar{x}, \sigma_f^{n_i}(\bar{x}))$ is a γ_3 -quasi-expanding string of F , i.e.,

$$\frac{1}{\ell} \sum_{j=1}^{\ell} \log \|F_{\sigma_f^{n_i-j}(\bar{x})}\|_{\text{co}} \geq \gamma_3 \quad \forall \ell = 1, \dots, n_i \text{ for } 1 \leq i \leq k.$$

Then, for all $1 \leq i < k$, either $n_{i+1} - n_i \leq N_{\bar{\gamma}_2, \gamma_3}$ or $(\sigma_f^{n_i}(\bar{x}), \sigma_f^{n_{i+1}}(\bar{x}))$ is an $(N_{\bar{\gamma}_2, \gamma_3}, \bar{\gamma}_2)$ -obstruction of F . Here $N_{\bar{\gamma}_2, \gamma_3}$ is defined in the manner as in Lemma 2.2 with constants $\gamma = \bar{\gamma}_2$, $\gamma' = \gamma_3$ and $H = \max \{|\log \|D_x f\|_{\text{co}}|; x \in M^n\}$.

The following result is a special case of [25, Lemma II-5] in the case of $r = 0$.

Lemma 3.4 ([25, Lemma II-5]). *Let there be any given real numbers*

$$\gamma_0 > \gamma_1 > \gamma_2 > \gamma_3 > 0$$

and integers

$$m > \ell > n > 0.$$

Let $(\bar{x}, \sigma_f^m(\bar{x}))$ be a γ_0 -string of F and $(\bar{x}, \sigma_f^\ell(\bar{x}))$ an (n, γ_2) -obstruction of F . Assume

- (a) $m \geq N_{\gamma_0, \gamma_3}$,
- (b) $mc_{\gamma_0, \gamma_3} > \ell$,
- (c) $\ell \geq N_{\gamma_1, \gamma_2}$ and
- (d) $\ell c_{\gamma_1, \gamma_2} > n$.

Then, there exists a γ_3 -quasi-expanding string $(\bar{x}, \sigma_f^k(\bar{x}))$ of F with $\ell \leq k < m$, such that $(\bar{x}, \sigma_f^k(\bar{x}))$ is not a γ_1 -string of F . Here all N_{γ_0, γ_3} , c_{γ_0, γ_3} and N_{γ_1, γ_2} , c_{γ_1, γ_2} are defined in the manner as in Lemma 2.2 with $H = \max \{|\log \|D_x f\|_{\text{co}}|; x \in M^n\}$.

Let $\Theta \subset \Sigma_f$ be a forward invariant non-void closed set of σ_f and $\bar{x} \in \Theta$. Following [25], we define the “germ” as follows:

$$J(\bar{x}, \Theta) = \left\{ \bar{y} \in \Theta \mid \exists \{x_k\} \text{ and } \{m_k\} \text{ such that } \bar{y} = \lim_{k \rightarrow +\infty} \sigma_f^{m_k}(\bar{x}_k) \right\}$$

where $\{\bar{x}_k\}$ is a sequence in Θ converging to \bar{x} and $m_k \rightarrow +\infty$.

Clearly to obtain $J(\bar{x}, \Theta)$, it is sufficient to use sequences $\{\bar{x}_k\}$ contained in a dense subset Θ_0 of Θ . Since m_k converges to $+\infty$ as $k \rightarrow +\infty$, it is easily seen that $J(\bar{x}, \Theta)$ is closed and σ_f -invariant. Moreover, if $\Theta = \Omega(\sigma_f|_{\Theta})$ the nonwandering point set of the restriction of σ_f to Θ , then \bar{x} itself belongs to $J(\bar{x}, \Theta)$. In addition, $J(\bar{x}, \Theta)$ is nonempty because every ω -limit point of \bar{x} belongs to it.

The following notion is a modification of Mañé’s “ (t, γ) -set” defined there for a diffeomorphism [25, pp. 177].

Definition 3.5. A forward σ_f -invariant compact subset Σ' of Σ_f is called a (t, γ) -set of F where $t \in \mathbb{N}$ and $\gamma > 0$, provided that for every $\bar{z} \in \Sigma'$, there exists an integer $m_{\bar{z}} \in [0, t)$ such that $(\sigma_f^{m_{\bar{z}}-r}(\bar{z}), \sigma_f^r(\sigma_f^{m_{\bar{z}}-r}(\bar{z})))$ is a γ -string of F for all $r > 0$.

Clearly, this implies that F is expanding on Σ' . Note here that we do not require $\sigma_f^{m_\varepsilon-r}(\bar{z})$ to be in Σ' , since Σ' is only forwardly invariant.

The following lemma will be needed in the proof of our Theorem 1.

Lemma 3.6. *Let there be given a forward σ_f -invariant compact set $\Theta \subset \Sigma_f$, on which F is not expanding. Assume there is the number $c > 0$ for which there holds the inequality*

$$\limsup_{k \rightarrow +\infty} \frac{1}{k} \sum_{i=0}^{k-1} \log \|F_{\sigma_f^i(\bar{x})}\|_{\text{co}} \geq c$$

for a dense set Θ_0 of points $\bar{x} \in \Theta$. Then, for any $\epsilon > 0$ and $c > \gamma_2 > \bar{\gamma}_2 > \gamma_3 > 0$, there exists an integer $N = N_{\epsilon, \gamma_2, \bar{\gamma}_2, \gamma_3} \geq 1$ such that for any $\bar{x} \in \Theta$,

- (1) either $J(\bar{x}, \Theta)$ is an (N, γ_3) -set of F ;
- (2) or there exists $\bar{y} \in \Theta$ such that $(\bar{y}, \sigma_f^m(\bar{y}))$ is an $(N_{\bar{\gamma}_2, \gamma_3}, \gamma_2)$ -obstruction of F for all $m \geq N_{\bar{\gamma}_2, \gamma_3}$, where $N_{\bar{\gamma}_2, \gamma_3}$ is given by Lemma 2.2 with $H = \max_{x \in M^n} |\log \|D_x f\|_{\text{co}}|$; moreover, such \bar{y} satisfies at least one of the following properties:
 - a) $d_f(\bar{x}, \bar{y}) \leq \epsilon$;
 - b) there exists $\bar{u} \in \Theta$ arbitrarily near to \bar{x} and $m \geq 1$ such that $d_f(\sigma_f^m(\bar{u}), \bar{y}) < \epsilon$ and that $(\bar{u}, \sigma_f^m(\bar{u}))$ is a γ_3 -quasi-expanding string of F .

Proof. The argument is a modification of that of [25, Lemma II-6]. We denote by Θ' the set of points $\bar{y} \in \Theta$ such that $(\bar{y}, \sigma_f^m(\bar{y}))$ is an $(N_{\bar{\gamma}_2, \gamma_3}, \gamma_2)$ -obstruction of F for all $m \geq N_{\bar{\gamma}_2, \gamma_3}$. Since F is not expanding on Θ by hypothesis, the set Θ' is obviously non-void.

It is easy to check that there exists $N = N_{\epsilon, \gamma_2, \bar{\gamma}_2, \gamma_3} > N_{\bar{\gamma}_2, \gamma_3}$ such that when $(\bar{y}, \sigma_f^m(\bar{y}))$ is an $(N_{\bar{\gamma}_2, \gamma_3}, \bar{\gamma}_2)$ -obstruction of F in Θ for some $m > N$, then $d_f(\bar{y}, \Theta') < \epsilon$. Here we have used the hypothesis $\gamma_2 > \bar{\gamma}_2$, the C^1 -smoothness of f , and the compactness of Θ .

Given any $\bar{x} \in \Theta$ and any $\bar{z} \in J(\bar{x}, \Theta)$, there exists a sequence $\{\bar{x}_k\}_{k \geq 1}$ in Θ_0 converging to \bar{x} and satisfying $\bar{z} = \lim_{k \rightarrow +\infty} \sigma_f^{m_k}(\bar{x}_k)$, where m_k converges to $+\infty$ as k tends to $+\infty$. For any $k \geq 1$, define the set of integers

$$\mathcal{S}_k = \{m > 0 \mid (\bar{x}_k, \sigma_f^m(\bar{x}_k)) \text{ is a } \gamma_3\text{-quasi-expanding string of } F\} \cup \{0\}.$$

As $\gamma_3 < c$ and $\bar{x}_k \in \Theta_0$, from Lemma 2.2 it follows easily that \mathcal{S}_k is infinite. For any $k \geq 1$, set

$$k^+ = \min \mathcal{S}_k \cap [m_k, +\infty) \quad \text{and} \quad k^- = \max \mathcal{S}_k \cap [0, m_k).$$

Suppose that $\liminf_{k \rightarrow +\infty} (k^+ - k^-) \leq N$. Then, there exists some integer $m_\varepsilon \in [0, N]$ which satisfies that $\sigma_f^{m_\varepsilon}(\bar{z})$ is the limit of some subsequence of $\{\sigma_f^{k^+}(\bar{x}_k) \mid k \geq 1\}$. Hence, $(\sigma_f^{m_\varepsilon-r}(\bar{z}), \sigma_f^{m_\varepsilon}(\bar{z}))$ is a γ_3 -string of F for all $r \geq 1$ (here we use the property $m_k \rightarrow +\infty$). If this holds for all $\bar{z} \in J(\bar{x}, \Theta)$ then, $J(\bar{x}, \Theta)$ is an (N, γ_3) -set of F . If this is not the case, the above argument shows that we can always pick some $\bar{z} \in J(\bar{x}, \Theta)$ such that for any sufficiently large k , $k^+ - k^- > N$. Hence $k^+ - k^- > N_{\bar{\gamma}_2, \gamma_3}$ because of $N > N_{\bar{\gamma}_2, \gamma_3}$. Then, by Lemmas 2.2 and 3.3, it follows that $(\sigma_f^{k^-}(\bar{x}_k), \sigma_f^{k^+}(\bar{x}_k))$ is an $(N_{\bar{\gamma}_2, \gamma_3}, \bar{\gamma}_2)$ -obstruction of F in Θ . So, $d_f(\sigma_f^{k^-}(\bar{x}_k), \Theta') < \epsilon$ for sufficiently large k . If $k^- = 0$ for all sufficiently large k that satisfy $d_f(\sigma_f^{k^-}(\bar{x}_k), \Theta') < \epsilon$, then

$d_f(\bar{x}_k, \Theta') < \epsilon$ and since $\bar{x}_k \rightarrow \bar{x}$ we obtain $d_f(\bar{x}, \Theta') \leq \epsilon$. Taking $\bar{y} \in \Theta'$ such that $d_f(\bar{x}, \bar{y}) \leq \epsilon$, it follows that \bar{y} satisfies Lemma 3.6 and meanwhile the stipulation **a**). On the other hand, if for an unbounded set of k we have $k^- > 0$, we can take $\bar{y} \in \Theta'$ such that $d_f(\sigma_f^{k^-}(\bar{x}_k), \bar{y}) < \epsilon$ and then this point \bar{y} , the point $\bar{u} = \bar{x}_k$ and $m = k^-$ satisfy the requirements of Lemma 3.6 and the item **b**).

This completes the proof of Lemma 3.6. \square

4. A sifting-shadowing combination

This section is devoted to the most important argument of the sifting-shadowing combination for the proof of our main result Theorem 1.

By a sifting-shadowing combination, we will obtain the following criterion, which is of independent intrinsic interest.

Theorem 4.1. *Let there be given the forward invariant subset $\Lambda \subset M^n$ of a C^1 -class endomorphism $f: M^n \rightarrow M^n$, whose closure $\text{Cl}_{M^n}(\Lambda)$ in M^n does not contain any critical points of f . Assume there is the number $c > 0$ for which there holds the inequality:*

$$\limsup_{k \rightarrow +\infty} \frac{1}{k} \sum_{i=0}^{k-1} \log \|D_{f^i(x)} f\|_{\text{co}} \geq c \quad \forall x \in \Lambda.$$

If $\Lambda \subseteq \text{Per}(f)$ then, either f is expanding on $\text{Cl}_{M^n}(\Lambda)$ or for every neighborhood V of Λ in M^n and every $0 < \gamma' < \gamma'' < c$, there can be found in V a periodic orbit P of f with arbitrarily large period τ_P and satisfying the following “abnormal inequality” property:

$$\frac{1}{\tau_P} \sum_{i=0}^{\tau_P-1} \log \|D_{f^i(p)} f\|_{\text{co}} < \gamma''$$

and

$$\frac{1}{k} \sum_{i=1}^k \log \|D_{f^{\tau_P-i}(p)} f\|_{\text{co}} > \gamma' \quad \forall k = 1, \dots, \tau_P,$$

for some point $p \in P$.

We are going to prove this theorem following the framework of the proof of Mañé [25, Theorem II-1] that was clarified independently by [26, 29]. Let $d(\cdot, \cdot)$ be a metric on M^n compatible with the natural norm $\|\cdot\|$ on TM^n . The η -neighborhood of a set $X \subset M^n$, denoted by $B_\eta(X)$, is the union of the η -balls $B_\eta(x)$ around the points $x \in X$.

Proof. Since f does not have any critical points in $\text{Cl}_{M^n}(\Lambda)$ and it is of class C^1 , f is C^1 -class locally diffeomorphic restricted to a closed neighborhood of Λ . For simplicity, there is no loss of generality in assuming that f is C^1 -class locally diffeomorphic on the whole manifold M^n .

Let $\Lambda \neq \emptyset$. If f is expanding on $\text{Cl}_{M^n}(\Lambda)$, we may stop proving here. Now we assume f is not uniformly expanding on Λ .

To prove Theorem 4.1, we consider the natural extensions

$$\sigma_f: \Sigma_f \rightarrow \Sigma_f \quad \text{and} \quad F: T\Sigma_f \rightarrow T\Sigma_f$$

associated to f defined as in Section 3.1. Write $\Theta_0 = \Lambda_f$ and $\Theta = \text{Cl}_{\Sigma_f}(\Lambda_f)$, where Λ_f is the natural extension of Λ under f defined in the way as in Section 3.1. So, by the hypothesis of Theorem 4.1, F is not expanding on Θ such that

$$\limsup_{k \rightarrow +\infty} \frac{1}{k} \sum_{i=0}^{k-1} \log \|F_{\sigma_f^i(\bar{x})}\|_{\text{co}} \geq c \quad \forall \bar{x} \in \Theta_0,$$

where c is the constant given in the statement of Theorem 4.1. Moreover, $\Theta = \Omega(\sigma_f|\Theta)$, the nonwandering set of σ_f restricted to Θ .

Let $0 < \gamma' < \gamma'' < c$ be as in the statement of Theorem 4.1. Then, from now on we fix any γ_0 with $\gamma' < \gamma_0 < \gamma''$. To any $\bar{x} \in \Theta_0$, there can be found a sequence of positive integers $n_j(\bar{x}) \uparrow +\infty$ satisfying

$$\frac{1}{n_j(\bar{x})} \sum_{i=0}^{n_j(\bar{x})-1} \log \|F_{\sigma_f^i(\bar{x})}\|_{\text{co}} > \gamma_0 \quad \forall j = 1, 2, \dots$$

Let $\bar{\gamma}$ and $\hat{\gamma}$ be two arbitrary numbers such that $\gamma_0 > \bar{\gamma} > \hat{\gamma} > \gamma'$. Choose $\eta \in (0, 1)$ sufficiently small satisfying

$$\hat{\gamma} - \eta > \gamma' \quad \text{and} \quad \bar{\gamma} + \eta < \gamma''.$$

Take $\varepsilon > 0$ so small that

$$\hat{\gamma} - \varepsilon > \gamma', \quad B_{2\varepsilon}(\Lambda) \subset V$$

and that if $d(x, y) \leq \varepsilon$ for any $x, y \in M^n$ then

$$|\log \|D_x f\|_{\text{co}} - \log \|D_y f\|_{\text{co}}| < \eta.$$

Let $\delta = \delta(\varepsilon, \hat{\gamma})$ be given as in the statement of Theorem 2.1 with $\lambda = \hat{\gamma}$.

From the compactness of Σ_f , we can choose the positive integer $s = s(\delta/4)$ which satisfies that for any given sequence $\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_s\}$ in Σ_f there always can be found i, j with $1 \leq i \neq j \leq s$ such that $d_f(\bar{x}_i, \bar{x}_j) < \delta/4$. Now, we define $4(s+1)$ positive numbers as follows:

$$\left\{ \underbrace{\gamma_1^{(i)}, \gamma_2^{(i)}, \bar{\gamma}_2^{(i)}, \gamma_3^{(i)}}_{i=0, 1, \dots, s} \right\}$$

such that

$$\begin{aligned} \hat{\gamma} &= \underbrace{\gamma_3^{(0)} < \bar{\gamma}_2^{(0)} < \gamma_2^{(0)} < \gamma_1^{(0)}}_{i=0} < \underbrace{\gamma_3^{(1)} < \bar{\gamma}_2^{(1)} < \gamma_2^{(1)} < \gamma_1^{(1)}}_{i=1} < \dots < \gamma_1^{(i-1)} \\ &< \underbrace{\gamma_3^{(i)} < \bar{\gamma}_2^{(i)} < \gamma_2^{(i)} < \gamma_1^{(i)}}_{i=i} < \gamma_3^{(i+1)} < \dots < \gamma_1^{(s-1)} \\ &< \underbrace{\gamma_3^{(s)} < \bar{\gamma}_2^{(s)} < \gamma_2^{(s)} < \gamma_1^{(s)}}_{i=s} = \bar{\gamma}. \end{aligned}$$

Let

$$\mathbf{n}_i = N_{\frac{\delta}{4}; \gamma_2^{(i)}, \bar{\gamma}_2^{(i)}, \gamma_3^{(i)}} > N_{\bar{\gamma}_2^{(i)}, \gamma_3^{(i)}}$$

be the constants determined by Lemma 3.6 in the case of letting $\epsilon = \frac{\delta}{4}$, $\gamma_2 = \gamma_2^{(i)}$, $\bar{\gamma}_2 = \bar{\gamma}_2^{(i)}$ and $\gamma_3 = \gamma_3^{(i)}$ for all $0 \leq i \leq s$.

For any $i = 0, \dots, s$, let Σ_i be the compact set which consists of points $\bar{z} \in \Theta$ such that there exists an integer $m_{\bar{z}} \in [0, n_i]$ verifying that $(\sigma_f^{m_{\bar{z}}-r}(\bar{z}), \sigma_f^r(\sigma_f^{m_{\bar{z}}-r}(\bar{z})))$ is a $\gamma_3^{(i)}$ -string of F for all $r > 0$, i.e.,

$$\frac{1}{r} \sum_{i=0}^{r-1} \log \|F_{\sigma_f^i(\sigma_f^{m_{\bar{z}}-r}(\bar{z}))}\|_{\text{co}} \geq \gamma_3^{(i)} \quad \forall r > 0.$$

It need not be σ_f -invariant. Clearly, $\Theta \neq \bigcup_{i=0}^s \Sigma_i$, since F is not expanding on Θ .

For any $0 \leq i \leq s-1$, let us choose arbitrarily $\bar{x}_i \in \Theta \setminus \Sigma_i$. Then, $J(\bar{x}_i, \Theta)$ is not an $(n_i, \gamma_3^{(i)})$ -set of F because \bar{x}_i belongs to $J(\bar{x}_i, \Theta)$. From Lemma 3.6, there can be found $\bar{u}_i, \bar{y}_i \in \Theta$ and $m_i \geq 0$ such that $d_f(\bar{x}_i, \bar{u}_i) < \delta/4$, $d_f(\sigma_f^{m_i}(\bar{u}_i), \bar{y}_i) < \delta/4$ and $(\bar{u}_i, \sigma_f^{m_i}(\bar{u}_i))$ is a $\gamma_3^{(i)}$ -quasi-expanding string of F if $m_i > 0$, and that $(\bar{y}_i, \sigma_f^m(\bar{y}_i))$ is an $(N_{\gamma_2^{(i)}, \gamma_3^{(i)}}, \gamma_2^{(i)})$ -obstruction of F for all $m > n_i$. As Θ_0 is dense in Θ , it follows that when $\bar{z}_i \in \Theta_0$ is sufficiently close to \bar{y}_i , there exists a large $\ell > n_i$ such that $(\bar{z}_i, \sigma_f^\ell(\bar{z}_i))$ is an $(N_{\gamma_2^{(i)}, \gamma_3^{(i)}}, \gamma_2^{(i)})$ -obstruction of F . Moreover, there exist infinitely many m such that

$$\frac{1}{m} \sum_{j=0}^{m-1} \log \|F_{\sigma_f^j(\bar{z}_i)}\|_{\text{co}} > \gamma_0.$$

Applying Lemma 3.4 to $\gamma_1 = \gamma_1^{(i)}$, $\gamma_2 = \gamma_2^{(i)}$ and $\gamma_3 = \gamma_3^{(i)}$, because m can be chosen large with respect to ℓ , ℓ large with respect to n_i , there exists $m > k_i > \ell > n_i$ such that $(\bar{z}_i, \sigma_f^{k_i}(\bar{z}_i))$ is a $\gamma_3^{(i)}$ -quasi-expanding string of F but not a $\gamma_1^{(i)}$ -string of F , and so not a $\gamma_3^{(i+1)}$ -string of F too. Thus, $\sigma_f^{k_i}(\bar{z}_i)$ lies in $\Theta \setminus \Sigma_{i+1}$ when ℓ large enough. Further, by induction on i we can construct sequences $\{(\bar{u}_i, \bar{z}_i)\}_{i=0}^{s-1}$ and $\{(m_i, k_i)\}_{i=0}^{s-1}$ with $\bar{u}_i, \bar{z}_i \in \Theta$ and $m_i \geq 0, k_i \geq 2$, such that:

- 1) $(\bar{u}_i, \sigma_f^{m_i}(\bar{u}_i))$ and $(\bar{z}_i, \sigma_f^{k_i}(\bar{z}_i))$ both are $\gamma_3^{(i)}$ -quasi-expanding string of F , where $\bar{u}_i = \bar{z}_i$ if $m_i = 0$;
- 2) $(\bar{z}_i, \sigma_f^{k_i}(\bar{z}_i))$ is not a $\gamma_1^{(i+1)}$ -string of F ;
- 3) $d_f(\sigma_f^{m_i}(\bar{u}_i), \bar{z}_i) < \delta/2$;
- 4) $d_f(\sigma_f^{k_i}(\bar{z}_i), \bar{u}_{i+1}) < \delta/2$;
- 5) if $H = \max\{\log \|D_x f\|_{\text{co}}; x \in M^n\}$, then

$$k_i \gamma_1^{i+1} + m_i H \leq (m_i + k_i) \bar{\gamma}$$

for all $i = 0, \dots, s-1$.

In fact, because $\bar{\gamma} > \gamma_1^{(i)}$ for all i , we only need to take k_i sufficiently large in the previous construction to satisfy the conditions 1), 2), 3), 4), and 5) above.

By the definition of $s = s(\delta/4)$ before, there can be found in $\{\bar{u}_i\}_{i=0}^{s-1}$ two points \bar{u}_i, \bar{u}_{j+1} with $i < j$ such that $d_f(\bar{u}_i, \bar{u}_{j+1}) < \delta/4$. It is easy to check that the sequence

$$(\bar{u}_i, \sigma_f^{m_i}(\bar{u}_i)), (\bar{z}_i, \sigma_f^{k_i}(\bar{z}_i)), \dots, (\bar{u}_j, \sigma_f^{m_j}(\bar{u}_j)), (\bar{z}_j, \sigma_f^{k_j}(\bar{z}_j))$$

forms a periodic $\hat{\gamma}$ -quasi-expanding of F δ -pseudo-orbit of σ_f in Θ . Let

$$u_\ell = \pi(\bar{u}_\ell) \quad \text{and} \quad z_\ell = \pi(\bar{z}_\ell) \quad \text{for all } i \leq \ell \leq j,$$

where $\pi: \Sigma_f \rightarrow M^n$ is the natural projection defined as in Section 3.1. Then, by the definition of the metric function d_f of Σ_f it follows from Lemma 3.1 that the string

$$(u_i, f^{m_i}(u_i)), (z_i, f^{k_i}(z_i)), \dots, (u_j, f^{m_j}(u_j)), (z_j, f^{k_j}(z_j))$$

forms a periodic $\hat{\gamma}$ -quasi-expanding δ -pseudo-orbit of f in $\text{Cl}_{M^n}(\Lambda)$.

So, from Theorem 2.1 there can be found a periodic point p of f with period

$$\tau_p = m_i + k_i + \dots + m_j + k_j,$$

which ε -shadows the above δ -pseudo-orbit of f such that $\text{Orb}_f^+(p) \subset B_\varepsilon(\Lambda)$ and

$$\frac{1}{k} \sum_{i=1}^k \log \|D_{f^{\tau_p-i}(p)} f\|_{\text{co}} \geq \hat{\gamma} - \varepsilon > \gamma'$$

for all $k = 1, \dots, \tau_p$.

Since k_i can be chosen arbitrarily large, τ_p can also be arbitrarily large. The rest is to check that such p satisfies the abnormal inequality.

In fact, for all $i \leq \ell \leq j$, by **2)** and **5)** above we have

$$\begin{aligned} \sum_{t=0}^{m_\ell-1} \log \|D_{f^t(u_\ell)} f\|_{\text{co}} + \sum_{t=0}^{k_\ell-1} \log \|D_{f^t(z_\ell)} f\|_{\text{co}} &\leq m_\ell H + k_\ell \gamma_1^{(\ell+1)} \\ &\leq (m_\ell + k_\ell)(\eta + \bar{\gamma}). \end{aligned}$$

Thus,

$$\sum_{\ell=i}^j \left\{ \sum_{t=0}^{m_\ell-1} \log \|D_{f^t(u_\ell)} f\|_{\text{co}} + \sum_{t=0}^{k_\ell-1} \log \|D_{f^t(z_\ell)} f\|_{\text{co}} \right\} \leq \tau_p \bar{\gamma}.$$

Because p ε -shadows this quasi-expanding pseudo-orbit string, we obtain that

$$\sum_{t=0}^{\tau_p-1} \log \|D_{f^t(p)} f\|_{\text{co}} \leq \tau_p(\eta + \bar{\gamma}).$$

Thus

$$\frac{1}{\tau_p} \sum_{t=0}^{\tau_p-1} \log \|D_{f^t(p)} f\|_{\text{co}} \leq \eta + \bar{\gamma} < \gamma''.$$

This ends the proof of Theorem 4.1. □

5. Proof of Theorem 1 and local diffeomorphisms of the circle

In this section, we will prove Theorem 1 using the theorems proved before. Then Theorem 2 follows easily from Theorem 1.

5.1. Proof of Theorem 1

The first statement of Theorem 1 follows immediately from Theorem 4.1 with $\Lambda = \text{Per}(f)$ and $c = \lambda$.

The second part of Theorem 1 comes from Theorem 2.3. In fact, if $\lambda_{\min}(\mu, f) > 0$ then from Theorem 2.3, it follows that $\text{supp}(\mu) \subseteq \text{Cl}_{M^n}(\text{Per}(f))$; and so from the first part of Theorem 1 proved, we can obtain that

$$\begin{aligned} \lambda_{\min}(\mu, f) &= \lim_{k \rightarrow +\infty} \frac{1}{k} \log \|D_x f^k\|_{\text{co}} \quad \mu\text{-a.e. } x \in \text{Cl}_{M^n}(\text{Per}(f)) \\ &= \lim_{k \rightarrow +\infty} \frac{1}{k} \log \min \{ \|(D_x f^k)v\|; v \in T_x M^n \text{ and } \|v\| = 1 \} \\ &\geq \lim_{k \rightarrow +\infty} \frac{1}{k} \log(C \exp(k\lambda)) \\ &= \lambda. \end{aligned}$$

And the third part of Theorem 1 trivially holds from the first statement of this theorem. In fact, it follows from statement (1) of Theorem 1 that f is uniformly expanding on $\Omega(f)$. As all ergodic measures of f are supported on $\Omega(f)$, it follows from Mañé's criterion [25, Lemma I-5] as mentioned in Section 1.3 that there exists an integer $m \geq 1$ and a constant $\lambda' > 0$ such that

$$\int_{M^n} \log \|D_x f^m\|_{\text{co}} d\mu \geq \lambda'$$

for all ergodic measures μ of f supported on M^n . Thus, f is uniformly expanding on M^n from Mañé's criterion once again.

This completes the proof of Theorem 1.

5.2. Proof of Theorem 2

Let

$$T_x M^n = E(x) \oplus F(x) \quad \text{with } \dim E(x) = 1 \quad \forall x \in \text{Cl}_{M^n}(\text{Per}(f))$$

be an $(\eta, 1)$ -dominated splitting given by the hypotheses of Theorem 2. Then, there exists an integer $m \geq 1$ such that

$$\frac{\|D_x f^m|E(x)\|}{\|D_x f^m|F(x)\|_{\text{co}}} \leq \frac{1}{2} \quad \forall x \in \text{Cl}_{M^n}(\text{Per}(f)).$$

As the minimal Lyapunov exponent $\lambda_{\min}(x, f) > 0$ and is uniformly bounded away from 0 for $x \in \text{Per}(f)$, from $\dim E(x) = 1$ it follows that

$$\lambda_{\min}(x, f^m) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\ell=0}^{N-1} \log \|D_{f^{\ell m}(x)} f^m|E\| = m\lambda_{\min}(x, f) \geq \lambda$$

for some constant $\lambda > 0$. Hence $Df^m|E$ and then Df^m are nonuniformly expanding on $\text{Per}(f^m)$. Thus Theorem 1 implies that f^m is uniformly expanding on $\text{Cl}_{M^n}(\text{Per}(f^m))$.

This completes the proof of Theorem 2.

5.3. Local diffeomorphisms of the circle

As another application of our result Theorem 1, we will consider a local diffeomorphism of the unit circle in this subsection.

Let \mathbb{T}^1 be the unit circle. Using Theorem 1 we can obtain the following result, which is indeed a special case of Theorem 2.

Theorem 3. *Let $f: \mathbb{T}^1 \rightarrow \mathbb{T}^1$ be a C^1 -class endomorphism of \mathbb{T}^1 which does not contain any critical points. If every periodic point of f has a positive Lyapunov exponent and such exponent is uniformly bounded away from zero, then f is uniformly expanding on $\text{Cl}_{\mathbb{T}^1}(\text{Per}(f))$ and moreover, for any ergodic measure μ of f , either its support $\text{supp}(\mu)$ is contained in $\text{Cl}_{\mathbb{T}^1}(\text{Per}(f))$ or its Lyapunov exponent is zero.*

Proof. First, we assume $\text{Per}(f) \neq \emptyset$. Then $\text{Per}(f)$ is nonuniformly expanding by f and then the statement comes immediately from Theorems 1 and 2.3. We notice that from [20], it follows that for any ergodic measure μ of f , its Lyapunov exponent $\lambda(\mu) \geq 0$.

If $\text{Per}(f) = \emptyset$, then from Theorem 2.3 we see that for any ergodic measure μ of f , its Lyapunov exponent must be zero.

This proves Theorem 3. \square

We here give a remark on the proof of Theorem 3 above. It is known, from [9], that in the 1-dimensional case Lyapunov exponent is continuous with respect to ergodic measures in the sense of weak-* topology. However, although $\text{Per}(f)$ is dense in $\text{Cl}_{\mathbb{T}^1}(\text{Per}(f))$, one still cannot guarantee, without any generic condition, that every ergodic measure of f supported on $\text{Cl}_{\mathbb{T}^1}(\text{Per}(f))$ can be arbitrarily approximated by periodic measures. So, the proof of Theorem 3 presented above is of interest itself.

In the situation of Theorem 3, if f does not have any periodic points, then from Theorem 3 we see that

$$|f'_{f^n(x)} \cdots f'_x|^{1/n} \rightarrow 1 \quad \text{as } n \rightarrow +\infty$$

uniformly for $x \in \mathbb{T}^1$.

6. Appendix: closing up quasi-expanding strings

In the section, we will prove Theorem 2.1 stated in Section 2.1 following the standard way, see Gan [15], for example.

For this, we need a simple sequence version of shadowing lemma borrowed from [15]. In the following lemma, we let $Y = \{(x_i)_{i \in \mathbb{Z}} \mid x_i \in X_i\}$ where X_i is an n -dimensional Euclidean space endowed with the norm $\|\cdot\|_i$ for every i . Under the supremum norm $\|y\| = \sup_{i \in \mathbb{Z}} \|x_i\|_i$ for $y = (x_i)$, Y is a Banach space. We only consider the mapping $\Phi: Y \rightarrow Y$ which has the form

$$(\Phi y)_{i+1} = \Phi_i x_i \quad \text{where } \Phi_i: X_i \rightarrow X_{i+1} \quad \forall i \in \mathbb{Z}.$$

For any $r > 0$, let $X_i(r) = \{x_i \in X_i; \|x_i\|_i \leq r\}$.

Now, the sequence version of shadowing lemma for expanding pseudo-orbit can be described as follows:

Lemma 6.1. *Let there be given numbers $\gamma \in (0, 1)$, $\epsilon > 0$ with*

$$\epsilon_1 := 2\epsilon(1 + \gamma)/(1 - \gamma) < 1$$

and $r > 0$. Let $\varsigma = \frac{(1-\gamma)(1-\epsilon_1)}{2(1+\gamma)}$ and $\delta \in (0, r\varsigma]$. Assume $\Phi = (\Phi_i)_{i \in \mathbb{Z}}: Y \rightarrow Y$ has the form

$$\Phi_i = H_i + \phi_i: X_i(r) \rightarrow X_{i+1}$$

where $H_i: X_i \rightarrow X_{i+1}$ is a linear isomorphism. If there holds that

$$\|H_i\|_{\text{co}} \geq \gamma^{-1}, \quad \text{Lip}(\phi_i) \leq \varsigma, \quad \text{and} \quad \|\phi_i(\mathbf{0})\| \leq \delta \quad \forall i \in \mathbb{Z},$$

then Φ has a unique fixed point v in Y with $\|v\| \leq \delta\varsigma^{-1}$.

This statement is a simple consequence of Gan [15, Theorem 2.3]. So we omit its proof here.

Let $\gamma \in (0, 1)$ be arbitrarily given. A positive number string $(b_i)_{i=0}^{\ell-1}$ of length $\ell \geq 1$, is called γ -expanding if there holds $b_i \geq \gamma^{-1}$ for all $i = 0, \dots, \ell - 1$. It is called γ -quasi-expanding if there holds the condition: $\prod_{i=1}^k b_{\ell-i} \geq \gamma^{-k}$ for all $k = 1, \dots, \ell$.

A string of positive numbers $(c_i)_{i=0}^{\ell-1}$ is called well-adapted to a γ -quasi-expanding string $(b_i)_{i=0}^{\ell-1}$, provided that $\prod_{i=0}^{\ell-1} c_i = 1$ and $\prod_{i=0}^k c_i \leq 1$ for $k = 0, \dots, \ell - 2$ if $\ell \geq 2$ and $(b_i/c_i)_{i=0}^{\ell-1}$ is γ -expanding.

Then, the following is a special case of the combinatorial lemma of Liao [21], also see [15, 11].

Lemma 6.2. *Let $\gamma \in (0, 1)$ be arbitrarily given. Any γ -quasi-expanding string $(b_i)_{i=0}^{\ell-1}$ of length $\ell \geq 2$ has a well-adapted string $(c_i)_{i=0}^{\ell-1}$ such that $\min\{\gamma b_i, 1\} \leq c_i \leq b_i$ for all $0 \leq i < \ell$.*

The following lemma is standard:

Lemma 6.3. *Given any $f \in \text{Diff}_{\text{loc}}^1(M^n)$. For any $\varepsilon, \tau, \hat{\varsigma} > 0$ there exists a number r with $0 < r \leq \varepsilon$ such that if $x, y \in M^n$ satisfy $d(f(x), y) \leq r$, then the lift of f at (x, y)*

$$\Phi_{x \rightsquigarrow y} = \exp_y^{-1} \circ f \circ \exp_x: T_x M^n(r) \rightarrow T_y M^n$$

can be well defined such that $\Phi_{x \rightsquigarrow y} = H_{xy} + \phi_{xy}$ where $\text{Lip}(\phi_{xy}) \leq \hat{\varsigma}$ and where H_{xy} is a linear isomorphism satisfying

$$1 - \tau \leq \frac{\|H_{xy}\|_{\text{co}}}{\|D_x f\|_{\text{co}}} \leq 1 + \tau.$$

In what follows, let

$$K = \sup_{x \in M} \{ \|D_x f\|, \|(D_x f)^{-1}\| \}.$$

Now, we are ready to prove the theorem.

Proof of Theorem 2.1. We need to prove only the closing property. Let $\lambda > 0$ and $\varepsilon > 0$ be arbitrarily given as in the statement of Theorem 2.1.

Let $((x_i, f^{n_i}(x_i)))_{i=-\infty}^{+\infty}$ be a λ -quasi-expanding δ -pseudo-orbit of f in M^n where $\delta > 0$ be arbitrarily given; i.e., $(x_i, f^{n_i}(x_i))$ is a λ -quasi-expanding string of length n_i with $d(f^{n_i}(x_i), x_{i+1}) < \delta$ and $n_i \geq 1$ for each $i \in \mathbb{Z}$. Write

$$N_i = \begin{cases} 0 & \text{if } i = 0, \\ n_0 + n_1 + \dots + n_{i-1} & \text{if } i > 0, \\ -n_i - n_{i+1} - \dots - n_{-1} & \text{if } i < 0. \end{cases}$$

Let $y_j = f^{j-N_i}(x_i)$ and $X_j = T_{y_j}M^n$ for any $N_i \leq j < N_{i+1}$ and any $i \in \mathbb{Z}$. Then $(y_j)_{j \in \mathbb{Z}}$ is a δ -pseudo-orbit of f in M^n .

Next, we will ε -shadow $(y_j)_{j \in \mathbb{Z}}$ by a real orbit of f if δ is sufficiently small.

It is easily seen that there can be found two numbers $\tau \in (0, 1)$ and $\gamma \in (0, 1)$ such that

$$(1 - \tau) \exp \lambda \geq \gamma^{-1}.$$

We now take $\epsilon > 0$ small enough to satisfy

$$\epsilon_1 := \frac{2\epsilon(1 + \gamma)}{1 - \gamma} < 1.$$

Let

$$\varsigma = \frac{(1 - \gamma)(1 - \epsilon_1)}{2(1 + \gamma)} \quad \text{and} \quad \delta \in (0, r\varsigma]$$

where r is determined by Lemma 6.3 in correspondence with the triplet $(\varepsilon, \tau, \hat{\varsigma})$ where $\hat{\varsigma} = \varsigma/(K \exp \lambda)$. Then, according to Lemma 6.3 the lift $\Phi_{y_j \rightsquigarrow y_{j+1}}$ of f at (y_j, y_{j+1})

$$\Phi_j := \exp_{y_{j+1}}^{-1} \circ f \circ \exp_{y_j} : X_j(r) \rightarrow X_{j+1} \quad (N_i \leq j \leq N_{i+1} - 1)$$

has the form $\Phi_j = H_j + \phi_j$ such that

$$\|H_j\|_{\text{co}} \geq (1 - \tau)\|D_{y_j}f\|_{\text{co}}, \quad \text{Lip}(\phi_j) \leq \frac{\varsigma}{K \exp \lambda}$$

for $j = N_i, \dots, N_{i+1} - 1$, and

$$\phi_j(\mathbf{0}) = \mathbf{0} \quad \text{for } j = N_i, \dots, N_{i+1} - 2,$$

and

$$\|\phi_j(\mathbf{0})\| \leq \delta \quad \text{for } j = N_{i+1} - 1.$$

So, $(\|H_j\|_{\text{co}})_{j=N_i}^{N_{i+1}-1}$ is a γ -quasi-expanding string because the string $(\|D_{y_j}f\|_{\text{co}})_{j=N_i}^{N_{i+1}-1}$ is $e^{-\lambda}$ -quasi-expanding by the hypothesis of the theorem. And hence corresponding to it, there can be found from Lemma 6.2 a well-adapted string $(c_j)_{j=N_i}^{N_{i+1}-1}$ of length $N_{i+1} - N_i$ such that $K^{-1} \exp(-\lambda) \leq c_j \leq K$ for all $N_i \leq j \leq N_{i+1} - 1$.

For any $i \in \mathbb{Z}$ and any $N_i \leq j \leq N_{i+1} - 1$ let

$$g_j = \prod_{k=N_i}^j c_k, \quad \tilde{H}_j = c_j^{-1} H_j, \quad \tilde{\phi}_j(v) = g_j^{-1} \phi_j(g_{j-1}v),$$

and further define

$$\tilde{\Phi}_j = \tilde{H}_j + \tilde{\phi}_j : X_j \rightarrow X_{j+1}.$$

Denote by $\Psi_j = \Phi_j \cdots \Phi_{N_i}$ and $\tilde{\Psi}_j = \tilde{\Phi}_j \cdots \tilde{\Phi}_{N_i}$ for $N_i \leq j \leq N_{i+1} - 1$. Then we have $\tilde{\Psi}_j = g_j^{-1} \Psi_j$.

Note that $g_{N_{i+1}-1} = 1$ and $\tilde{\Psi}_{N_{i+1}-1} = \Psi_{N_{i+1}-1}$ for any i .

Thus, $\text{Lip}(\tilde{\phi}_j) = g_j^{-1} \text{Lip}(\phi_j g_{j-1}) = c_j^{-1} \text{Lip}(\phi_j) \leq \varsigma$ for all j and $\tilde{\phi}_j(\mathbf{0}) = \mathbf{0}$ for any $N_i \leq j < N_{i+1} - 1$ and $\|\tilde{\phi}_j(\mathbf{0})\| = \|\phi_j(\mathbf{0})\| \leq \delta$ for $j = N_{i+1} - 1$. Then, according to Lemma 6.1, $\tilde{\Phi} = (\tilde{\Phi}_j) : Y(r) \rightarrow Y$ where $Y = \prod_{j \in \mathbb{Z}} X_j$, has a unique fixed point $\tilde{v} = (\tilde{v}_j)$ such that $\|\tilde{v}\| \leq \delta \varsigma^{-1} < \varepsilon$.

Let $v_{N_i} = \tilde{v}_{N_i}$ for all i and we recursively define

$$v_j = \Phi_{j-1}(v_{j-1}) \quad \forall N_i < j < N_{i+1} - 1.$$

To ensure this, we need to check that $\|v_j\| \leq \delta\zeta^{-1}$, i.e. $v_{j-1} \in X_{j-1}(r)$. Indeed, since

$$v_j = \Psi_{j-1}(v_{N_i}) = g_{j-1}\tilde{\Psi}_{j-1}(v_{N_i}) = g_{j-1}\tilde{v}_j,$$

we have $\|v_j\| \leq \|\tilde{v}_j\| \leq \delta\zeta^{-1}$. From

$$v_{N_{i+1}} = \tilde{v}_{N_{i+1}} = \tilde{\Psi}_{N_{i+1}-1}(\tilde{v}_{N_i}) = \Psi_{N_{i+1}-1}(v_{N_i}) = \Phi_{N_{i+1}-1}(v_{N_{i+1}-1}),$$

we see that $v = (v_j)$ is a fixed point of $\Phi = (\Phi_j)$ and $\|v\| \leq \delta\zeta^{-1} < \varepsilon$. Let $z = \exp_{y_0}(v_0)$. Thus, z can $\delta\zeta^{-1}$ -shadow $\{y_j\}$.

Now let $((x_i, f^{m_i}(x_i)))_{-\infty}^{+\infty}$ be periodic, i.e., there is some $k \geq 0$ such that $x_{i+k+1} = x_i$ and $n_{i+k+1} = n_i$ for all i . Define \tilde{w} in the way $(\tilde{w})_j = (\tilde{v})_{N_{k+1}+j}$ for any $j \in \mathbb{Z}$. Since both \tilde{v} and \tilde{w} are fixed points of $\tilde{\Phi}$ in $Y(\delta\zeta^{-1})$, $\tilde{v} = \tilde{w}$ by the uniqueness. Thus, $v = w$ and further z has period N_{k+1} .

This ends the proof of Theorem 2.1. \square

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